# Reply by the Authors to J. J. Gribble

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## Introduction

IN his Comment, Gribble provides further evidence that "traditional gain and phase margins may be misleading indicators of robustness for SISO plant models having poles or zeros near the imaginary axis, because small movements of the poles and zeros can give rise to very large magnitude and phase errors on nearby portions of the imaginary axis."[1] An additional reason is that specific parameter variations may have little effect on the Nyquist plot at the frequencies used to determine nominal gain and phase margins, while having significant destabilizing effect at other frequencies.

## Discussion

Control systems should be designed for satisfactory stability robustness over the uncertainty range of plant parameters. Gain margin (GM) and phase margin (PM) have long been used as design metrics, presumably because satisfactory values produce robust controllers. However, the relationship between these stability margins and the effect of plant variations is indirect. A more direct measure of stability robustness is the probability of instability, $\mathbb{P}_i$, which is estimated by Monte Carlo simulation.

Ten compensators designed for the 1990 American Control Conference benchmark problem[2] were analyzed in Ref. 3. Their GM and PM were evaluated as predictors of the closed-loop $\mathbb{P}_i$ that occurs when uncertain parameters are varied within limits. The nominal stability margins gave little indication of the probability of instability for reasons discussed here.

The Nyquist stability criterion is described in numerous automatic control texts (e.g., Ref. 4). Given a scalar compensator $K(s)$ and a scalar plant $G(s)$, the Nyquist plot portrays the input-output amplitude ratio and phase angle of $K(j\omega)G(j\omega)$ as the driving frequency $\omega$ ranges from $-\infty$ to $+\infty$. Given a closed-loop stable system, the GM is the smallest loop-gain variation that forces the Nyquist plot to pass through the $(-1, 0)$ point. The PM describes the smallest phase angle change that causes instability. For gain or phase angle changes greater than GM or PM, the number of encirclements of the $(-1, 0)$ point changes, indicating instability.

Parameter variations cause frequency-dependent changes in open-loop transfer function amplitude ratio and phase angle. GM and PM are each evaluated at a single frequency and do not account for shape changes over a range of frequencies.

Stochastic robustness analysis estimates the probability that the closed-loop system could be unstable given expected probability distributions of uncertain parameters.[5] With many parameters, each of which can take many values, the most efficient way to estimate $\mathbb{P}_i$ is to perform a Monte Carlo evaluation. On each trial, parameter values are produced by random number generators, and the stability of the perturbed system is evaluated. The number of unstable cases divided by the total number of cases gives an estimate of $\mathbb{P}_i$. The confidence level of this estimate depends on the number of trials, is independent of the number of uncertain parameters, and is easily computed. The term $\mathbb{P}_i$ always has a binomial distribution regardless of the parametric distributions.[5]

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The benchmark problem[2] required that a robust single-input/single-output compensator be designed for a mass-spring-mass system ($m_1$, $k$, and $m_2$). The state contained four elements, the control affects the first mass, and the second mass's position is fed back for control. The plant had undamped eigenvalues at $[\pm j(k(m_1+m_2)/m_1m_2), 0, 0]$. The benchmark problem required an output settling time of 15 s after an impulsive disturbance on $m_2$, with limited actuator use and closed-loop robustness. Plant parameters were fixed but uncertain in the ranges $0.5 < k < 2$, $0.5 < m_1 < 1.5$, and $0.5 < m_2 < 1.5$. (This was problem 2 of Ref. 2; Ref. 1 refers to problem 1 of Ref. 2.)

$\mathbb{P}_i$ was estimated for the 10 compensators, assuming uniform distributions of the parameters $m_1$, $k$, and $m_2$. The Monte Carlo evaluations involved 20,000 trials for each compensator; for a probability estimate of 0.1, the 95% confidence interval would be $\pm 0.004$.[5] The compensators (designs A–J) ranged from second to eighth order and were synthesized using several techniques, including approximate loop-transfer recovery, $H_\infty$ minimization, nonlinear constrained optimization, and game theory.

GM ranged from 2.14 to 15.1 dB for the 10 controllers, and PM ranged from 17.5 to 58.7 deg (Ref. 3). With the assumed parameter variations, $\mathbb{P}_i$ ranged from 23.7 to 0.4%. The distribution of GM and PM for these compensators was not uniform; all but one design had margins below 5 dB and 30 deg (Fig. 1), with clustering about 2.5 dB and 25 deg. The most robust design (D) had the highest margins, whereas the least robust design (J) had the lowest, suggesting good correspondence between stability margins and robustness.

A more detailed analysis leads to a different conclusion.[6] Aside from the extremes, Figs. 2 and 3 show little correlation between the stability margins and $\mathbb{P}_i$. For example, the design with the second largest GM had higher likelihood of instability than five designs with lower GM (Fig. 2). The correlation of $\mathbb{P}_i$ with PM was not appreciably better.

We examine the Nyquist plots for the three controllers with lowest gain margins (A, E, and J). The Nyquist contours are plotted for nominal masses and two values of the spring constant $k$ (1 and 0.5). Compensator transfer functions are presented in Ref. 3.

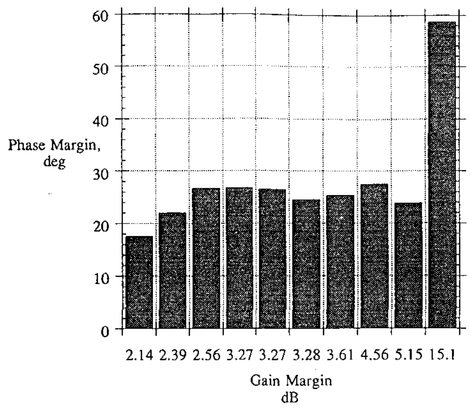


**Fig. 1 Relationship between GM and PM for 10 compensators.**

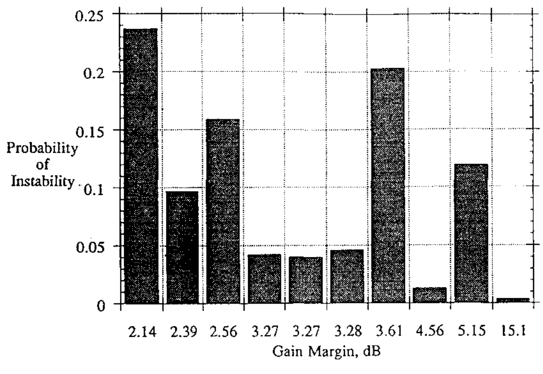


**Fig. 2 Probability of instability vs GM for 10 compensators.**

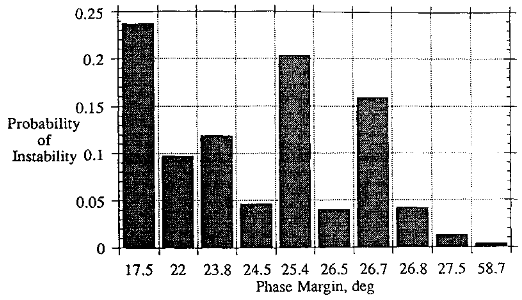


**Fig. 3 Probability of instability vs PM for 10 compensators.**

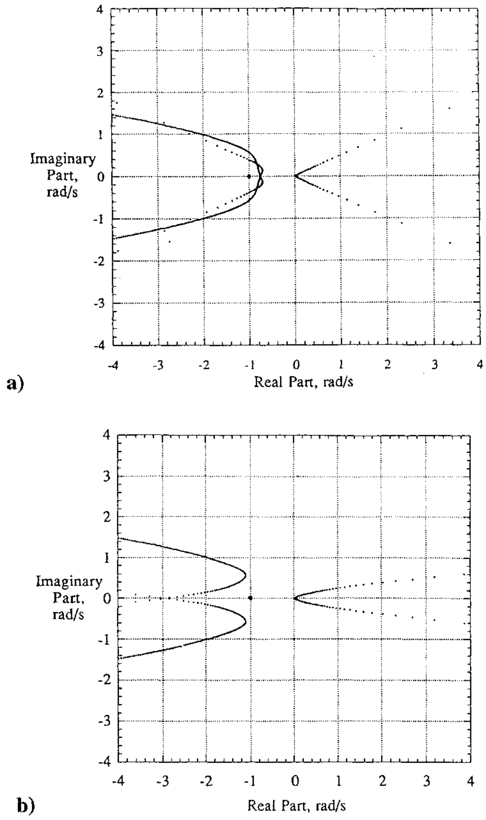


**Fig. 4 Nyquist plot for design A with two spring constant values: a) $k = 1$ and b) $k = 0.5$.**

Figures 4a and 4b show the effect of reducing $k$ for design A, whose nominal GM is 2.56 dB. A reduction of 50% in $k$ changes the number of encirclements of $(-1, 0)$, and the result is instability. Design E has lower gain margin (2.39 dB), yet the 50% reduction in $k$ does not change the number of encirclements (Figs. 5a and 5b). Design E has lower $\mathbb{P}_i$ than design A, suggesting that the shape of the Nyquist contour is important in this comparison.

The qualitative effects of variation in $k$ are similar for designs A and E, but they are considerably different for design J (Fig. 6). Design J has the highest $\mathbb{P}_i$, even though its GM is only marginally lower than that of design E. Design J also has higher order than the other two (sixth compared to third and second), so its Nyquist contour is more complex. In Fig. 6a, a low-frequency branch passes closest to the $(-1, 0)$ point; therefore, the stability margins are determined by this branch. When the spring constant is reduced (Fig. 6b), this branch remains relatively stationary, but a high-frequency branch of the contour moves past the low-frequency branch and past the $(-1, 0)$ point to cause instability. This effect could not be predicted by stability margins based on the nominal Nyquist contour.

Subsequent investigation has shown that highly robust compensators are readily designed using direct numerical search,[7] search efficiency can be dramatically improved using genetic algorithms,[8] and the stochastic method can be extended to nonlinear systems.[9]

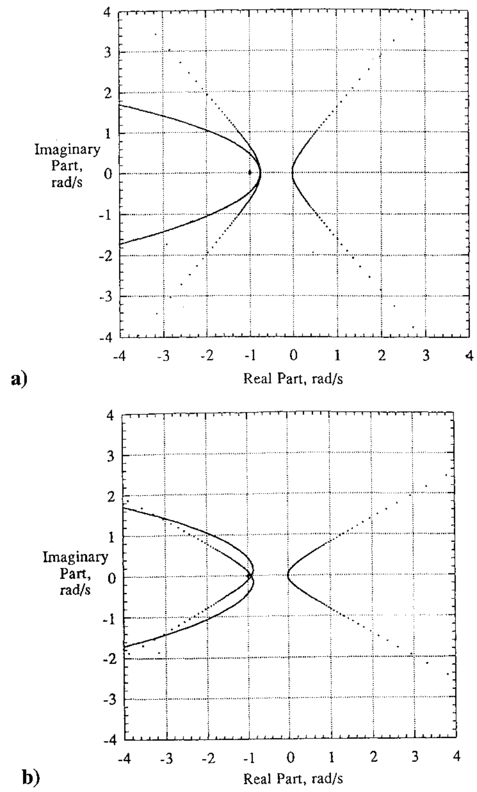


**Fig. 5 Nyquist plot for design E with two spring constant values: a) $k = 1$ and b) $k = 0.5$.**

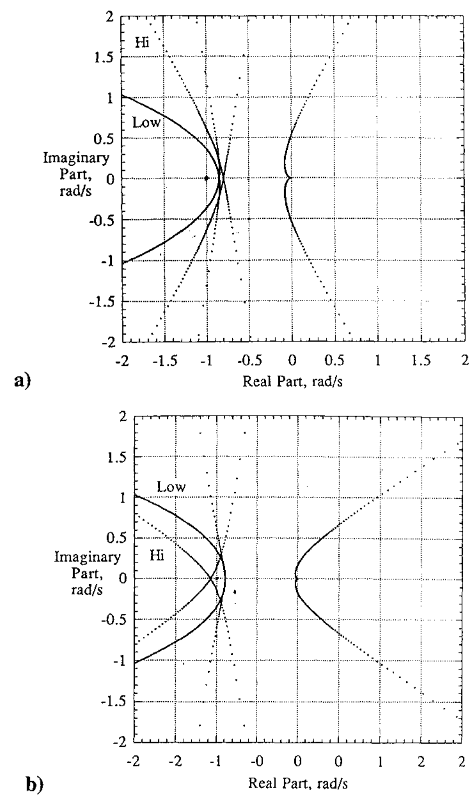


**Fig. 6 Nyquist plot for design J with two spring constant values: a) $k = 1$ and b) $k = 0.5$.**

## Conclusions

GM and PM characterize robustness by unspecified changes in the magnitude and phase angle of the nominal Nyquist plot. Real plant parameter variations can change the shape of the Nyquist plot as well. Stability margins are appropriate for comparing the robustness of compensators only if parametric effects leading to instability are relatively small and if the compensators themselves have similar structure. Stochastic robustness analysis provides more useful information about the effects of parameter uncertainty, and it leads to a practical approach for designing robust control systems.

## References

[1]Gribble, J. J., "Comment on 'Robustness of Solutions to a Benchmark Control Problem,'" *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 6, 1995, p. 1467.

[2]Wie, B., and Bernstein, D. S., "A Benchmark Problem for Robust Control Design," *Proceedings of the 1990 American Control Conference* (San Diego, CA), IEEE Press, Piscataway, NJ, 1990, pp. 961, 962.

[3]Stengel, R. F., and Marrison, C. I., "Robustness of Solutions to a Benchmark Control Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, 1992, pp. 1060–1067.

[4]Ogata, K., *Modern Control Engineering*, Prentice–Hall, Englewood Cliffs, NJ, 1990.

[5]Ray, L. R., and Stengel, R. F., "Stochastic Robustness of Linear-Time-Invariant Control Systems," *IEEE Transactions on Automatic Control*, Vol. 36, No. 1, 1991, pp. 82–87.

[6]Marrison, C. I., and Stengel, R. F., "Gain and Phase Margins as Indicators of Robustness," *Proceedings of the 1992 IEEE Regional Control Conference*, Polytechnic University, New York, 1992, pp. 5–8.

[7]Marrison, C. I., and Stengel, R. F., "Stochastic Robustness Synthesis Applied to a Benchmark Control Problem," *International Journal of Robust and Nonlinear Control*, Vol. 5, No. 1, 1995, pp. 13–31.

[8]Marrison, C. I., and Stengel, R. F., "The Use of Random Search and Genetic Algorithms to Optimize Stochastic Robustness Functions," *Proceedings of the 1994 American Control Conference* (Baltimore, MD), IEEE Press, Piscataway, NJ, 1994, pp. 1484–1489; see also *IEEE Transactions on Automatic Control* (to be published).

[9]Marrison, C. I., and Stengel, R. F., "Synthesis of Robust Control Systems for a Hypersonic Aircraft," *Proceedings of the 33rd IEEE Conference on Decision and Control* (Orlando, FL), IEEE Press, Piscataway, NJ, 1994, pp. 3324–3329.

# Errata

## Flight Test of Radar Altimeter Enhancement for Terrain-Referenced Guidance

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DURING the printing stage of this paper, Fig. 6b on page 706 was inadvertently dropped from the page. Here is the complete Fig. 6:

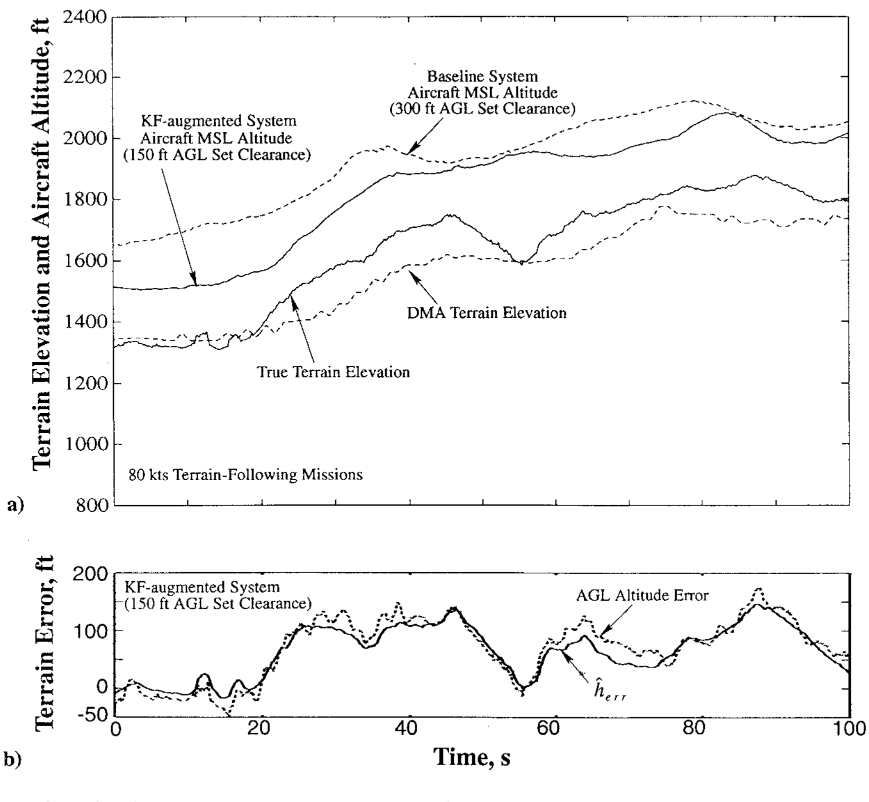


**Fig. 6 Baseline and Kalman-filter (KF-) augmented terrain-referenced guidance system during TF mission. Kalman-filter-augmented system allows for lower and more accurate flight above terrain than baseline system.**

AIAA regrets the error.